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Analysis of the Harmonic Raman–Nath equation

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Abstract. We analyse the Harmonic Raman–Nath equation and present a non-trivial perturbative solution. The connection with the conventional Raman–Nath equation is also discussed.

1. Introduction

A previous paper Bosco and Dattoli (1983) has been devoted to the analysis of the so called Raman–Nath (RN) equations Raman and Nath (1936) and was motivated by the fact that although they appear in many fields of physics (for a partial list of references see Bosco and Dattoli (1983)) a systematic approach to their study is still lacking.

The RN type equations belong to the class of recursive differential ones and their analytical solution seems to be unobtainable in terms of known functions. In Bosco and Dattoli (1983), however, a perturbative approach to the problem, useful for the numerical analysis of many physical situations, was presented.

Together with the RN equation another type of equation often appears. This equation, called henceforth the Harmonic Raman–Nath (HRN) equation, presents a number of analogies with the RN one and will be analysed, in the present paper, by means of a generalisation of the technique proposed in Bosco and Dattoli (1983).

For our aim we recall that the RN equation can be written in its most general form as (Bosco and Dattoli 1983)

$$\mathrm{i}(\mathrm{d}C_l/\mathrm{d}\tau) = (\alpha+\mu l)lC_l + \Omega(C_{l+1}+C_{l-1}) \tag{1.1}$$

where l is an integer, and the initial conditions are set by

$$C_l(0) = \delta_{l,0} \tag{1.2}$$

furthermore α, μ and Ω are constants.

The HRN equation reads

$$\mathrm{i}(\mathrm{d}C_l/\mathrm{d}\tau) = (\alpha+\mu l)lC_l + \Omega[(l+1)^{1/2}C_{l+1} + \sqrt{l}C_{l-1}] \tag{1.3}$$

with the same initial conditions. The choice of the term harmonic to distinguish (1.3) from (1.1) is self-explanatory.

A list of the physical problems to which (1.3) is relevant might be quite lengthy, let us quote among them the free electron laser (Shore and Eberly 1976, Dattoli and

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Renieri 1984) and the interaction of a multilevel system with em radiation (Shore and Eberly 1976).

In this paper, however, we shall not consider a specific physical problem, but we shall be concerned with the analysis of the solution of (1.3) which may be of relevance to the above quoted problems.

The paper is organised as follows. In § 2 we shall present a method of solution for the 'unperturbed' limit, in § 3 we analyse the extension of the method to the first 'order-perturbed' solution, finally § 4 will be devoted to the discussion of the connection of the HRN equation with the RN one and to conclusive remarks.

2. Unperturbed limit

The exact solution of (1.3) is beyond the scope of this paper and, as already remarked in the introduction, perhaps not even obtainable in terms of known functions.

We shall content ourselves with a non-trivial perturbative solution using, as expansion parameter, the coefficient μ. The reason for this choice resides in the fact that many of the physical problems in which the HRN equation plays a significant role, share as a common feature the small relative size of μ with respect to the other parameters entering the equation.

The first step of our analysis will therefore be the solution of (1.3) in the 'unperturbed' limit, i.e. when μ is set zero. We concentrate on this case for two main reasons:

(i) Because an analytical solution is obtainable (Dattoli and Renieri 1983, 1984, Marcuse 1980).

(ii) Because the technique of solution employed turns our to be very effective for the perturbed case. The equation we are interested in is†

$$\begin{aligned} &\mathrm{i}\,\mathrm{d}C_l^0/\mathrm{d}\tau = \alpha l C_l^0 + \Omega[(l+1)^{1/2}C_{l+1}^0 + \sqrt{l}C_{l-1}^0] \\ &C_l^0(0) = \delta_{l,0} \end{aligned} \tag{2.1}$$

a solution of (2.1) can be found by a direct generalisation of the 'shift' operator technique employed in Bosco and Dattoli (1983).

We introduce two operators A and A^+ which once acting on any function of l give

$$Af_l = \sqrt{l}f_{l-1} \qquad A^+f_l = (l+1)^{1/2}f_{l+1}. \tag{2.2}$$

The 'annihilation' and 'creation' operators introduced above are different from the 'shift' ones of Bosco and Dattoli (1983) since they are not commuting quantities and fulfil the well known law of commutation

$$[A, A^+] = 1. \tag{2.3}$$

If we now set

$$C_l^0(x) = (-\mathrm{i})^l M_l^0(x)\exp(-\mathrm{i}\beta l x) \tag{2.4}$$

where $x = \Omega\tau$ and $\beta = \alpha/\Omega$, and we insert (2.4) into (2.1), we can rewrite our main equation, using (2.2), as

$$\begin{aligned} &\mathrm{d}M_l^0(x)/\mathrm{d}x = [\exp(\mathrm{i}\beta x)A - \exp(-\mathrm{i}\beta x)A^+]M_l^0(x) \\ &M_l^0(0) = \mathrm{i}^l\delta_{l,0}. \end{aligned} \tag{2.5}$$

† The superscript 0 stands for zero-order perturbation.

We can now write down the formal solution of (2.5) as follows

$$M_l^0(x) = \left[\exp\left(\int_0^x \hat{T}^0(x')\,\mathrm{d}x'\right)\right]_+ (\mathrm{i}^l\delta_{l,0}) \tag{2.6}$$

where $[,]_+$ denotes time ordering, indeed the 'transfer' operator

$$\hat{T}^0(x) = \exp(\mathrm{i}\beta x)A - \exp(-\mathrm{i}\beta x)A^+ \tag{2.7}$$

does not commute with itself at different 'times'.

We overcome the usual difficulties associated with the time ordering as in Bosco and Dattoli (1983) by making use of Magnus' Theorem (Magnus 1954), i.e. we shall employ the following expansion

$$\left[\exp\left(\int_0^x \mathrm{d}x'\hat{T}^0(x')\right)\right]_+ = \exp\left[\int_0^x \mathrm{d}x'\ \hat{T}^0(x') + \frac{1}{2}\int_0^x \mathrm{d}x'\left[\hat{T}^0(x'), \int_0^{x'} \hat{T}^0(x'')\,\mathrm{d}x''\right]\right] \tag{2.8}$$

for further comments see Bosco and Dattoli (1983) and references therein. We have expanded the exponents up to the first commutator since the successive ones vanish. In the next section we shall see how the structure of the expansion becomes more complicated.

We can therefore write explicitly

$$M_l^0(x) = \exp\left[\left(\frac{\sin\beta/2x}{\beta/2}\right)[\exp(\mathrm{i}\beta/2x)A - \exp(-\mathrm{i}\beta/2x)A^+]\right.$$
$$\left. -\mathrm{i}\beta/2\int_0^x \left(\frac{\sin(\beta x'/2)}{\beta/2}\right)^2 \mathrm{d}x'\right] i^l\delta_{l,0}. \tag{2.9}$$

We must now disentangle the exponents, this can be done by means of the dual of the Weyl–Barker–Hausdorff formula

$$\exp[A+B] = \exp[A]\exp[B]\exp(-1/2[A,B]) \tag{2.10}$$

which yields

$$M_l^0(x) = \exp\left[-\mathrm{i}\frac{\beta}{2}\int_0^x \left(\frac{\sin(\beta x'/2)}{\beta/2}\right)^2 \mathrm{d}x'\right]\exp\left[-\frac{1}{2}\left(\frac{\sin(\beta/2)x}{\beta/2}\right)^2\right]$$
$$\times\exp\left[\left(\frac{\sin(\beta/2)x}{\beta/2}\right)[-A^+\exp(-\mathrm{i}\beta x/2)]\right]$$
$$\times\exp\left[\left(\frac{\sin(\beta x/2)}{\beta/2}\right)[A\exp(\mathrm{i}\beta x/2)]\right]\mathrm{i}^l\delta_{l,0}$$
$$= \exp\left[-\mathrm{i}\frac{\beta}{2}\int_0^x \left(\frac{\sin(\beta x'/2)}{\beta/2}\right)^2 \mathrm{d}x'\right]\exp\left[-\frac{1}{2}\left(\frac{\sin(\beta x/2)}{\beta/2}\right)^2\right]$$
$$\times\left[\sum_{k=0}^\infty \frac{(-1)^k}{k!}\left(\frac{\sin(\beta x/2)}{\beta/2}\right)^k \exp[-\mathrm{i}k(\beta x/2)](A^+)^k\right]$$
$$\times\left[\sum_{n=0}^\infty \frac{1}{n!}\left(\frac{\sin(\beta x/2)}{\beta/2}\right)^n \exp[\mathrm{i}n(\beta x/2)]A^n\right]\mathrm{i}^l\delta_{l,0}. \tag{2.11}$$

From the above expression and from (2.4) we finally find

$$C_l^0(\tau) = (-\mathrm{i})^l \exp\left(-\mathrm{i}\frac{\alpha l}{2}\tau\right)\exp\left(-\mathrm{i}\frac{\alpha}{2}\int_0^\tau N(\tau')\,\mathrm{d}\tau'\right)P_l[N(\tau)] \tag{2.12}$$

where

$$N(\tau)=\Omega^2[\sin(\alpha\tau/2)/(\alpha/2)]^2 \tag{2.13}$$

and P_l is the 'Poisson' function defined as

$$P_l(x)=1/\sqrt{l!}x^{l/2}\,\mathrm{e}^{-x/2}. \tag{2.14}$$

We stress that the above result is by no means new, see e.g. a standard textbook such as Marcuse (1980). However, the operatorial technique we have developed turns out to be very effective in treating this kind of problem and may be easily generalised to more complicated cases as it will be seen in the next section.

3. First order perturbed solution

Let us now discuss the solution of (1.3) when μ is not zero, but small enough, with respect to the other quantities entering the equation, that a first order perturbed solution may work.

We can still use a transformation of the type (2.4), indeed by a natural generalisation we can set

$$C_l(x)=(-\mathrm{i})^l M_l(x)\exp[-\mathrm{i}(\beta+\rho l)lx] \tag{3.1}$$

where $\rho=\mu/\Omega$.

Inserting (3.1) into (1.3) we obtain the following equation

$$\begin{aligned}\mathrm{d}M_l(x)/\mathrm{d}x=&\sqrt{l}M_{l-1}(x)\exp\{\mathrm{i}[\beta+\rho(2l-1)]x\}\\&-(l+1)^{1/2}M_{l+1}(x)\exp\{-\mathrm{i}[\beta+\rho(2l+1)]x\}.\end{aligned} \tag{3.2}$$

The initial conditions are, of course, the same as (2.5).

We now rewrite (3.2), by means of the operators A and A^+, defined in the previous section, as

$$\mathrm{d}M_l(x)/\mathrm{d}x=[\![\exp\{\mathrm{i}[\beta+\rho(2A^+A+1)]x\}A-\exp\{-\mathrm{i}[\beta+\rho(2A^+A-1)]x\}A^+]\!]M_l(x). \tag{3.3}$$

By expanding the exponents up to the first order in ρ, (3.3) becomes

$$\mathrm{d}M_l(x)/\mathrm{d}x=\hat{T}(x)M_l(x) \tag{3.4}$$

where the 'transfer operator' $\hat{T}(x)$, is now written as

$$\begin{aligned}\hat{T}(x)=&-\cos\beta\tfrac{1}{2}xF_-+\mathrm{i}\sin\beta\tfrac{1}{2}xF_++2\rho(A^+A)(\partial/\partial\beta)(-\cos\beta\tfrac{1}{2}xF_-+\mathrm{i}\sin\beta\tfrac{1}{2}xF_+)\\&+\rho(\partial/\partial\beta)(\cos\beta\tfrac{1}{2}xF_+-\mathrm{i}\sin\beta\tfrac{1}{2}xF_-).\end{aligned} \tag{3.5}$$

The two operators $F_\pm$ just introduced, are defined as

$$F_\pm=A^+\exp(-\mathrm{i}\,\beta\tfrac{1}{2}x)\pm A\exp(\mathrm{i}\,\beta\tfrac{1}{2}x). \tag{3.6}$$

Equation (3.4) is only formally identical to (2.6) and the more complicated structure of the transfer operator $\hat{T}(x)$, will cause more and more terms to appear in Magnus' expansion.

A rehandling of Magnus' formula by Pechukas and Light (1966), has proved to be expedient for our purposes, namely

$$\left[\exp\left(\int_0^x dx'\, \hat{T}(x')\right)\right]_+$$
$$= \exp\left(\int_0^x dx'\, \hat{T}(x') - \tfrac{1}{2}\int_0^x dx' \int_0^{x'} dx'' [T(x'), T(x'')]\right.$$
$$+ \tfrac{1}{6}\int_0^x dx''' \int_0^{x'''} dx'' \int_0^{x''} dx' [\![[\hat{T}(x'), [\hat{T}(x''), \hat{T}(x''')]]$$
$$+ [[\hat{T}(x'), \hat{T}(x'')], \hat{T}(x''')]]\!]$$
$$- \tfrac{1}{12}\int_0^x dx'''' \int_0^{x''''} dx''' \int_0^{x'''} dx'' \int_0^{x''} dx'$$
$$\times ([\hat{T}(x'), [[\hat{T}(x''), \hat{T}(x''')], \hat{T}(x'''')]]$$
$$+ [[[\hat{T}(x'), [\hat{T}(x''), \hat{T}(x'''')]], \hat{T}(x''')]$$
$$\left. + [[\hat{T}(x'), \hat{T}(x'')], [\hat{T}(x'''), \hat{T}(x'''')]]) + \ldots \right) \tag{3.7}$$

where the dots stand for the higher terms, which turn out to be all vanishing, because the last commutator we consider gives rise to a commuting function (at least up to the first order in ρ).

The explicit evaluation of the exponent in (3.7) does not present any conceptual difficulty and reduces to a lengthy calculation of commutators and integrals.

The calculation machinery, however, can be in some sense simplified by taking advantage of the 'algebra' of the $F_\pm$ operators, whose commuting properties have been reported in the appendix.

After a tedious calculation we arrived at the following somewhat complex expression

$$M_l(x) = \exp\{\hat{L}_1(x) + \hat{L}_2(x) + \hat{L}_3(x) + \hat{L}_4(x)\}\, \mathrm{i}^l \delta_{l,0} \tag{3.8}$$

where

$$\hat{L}_1(x) = -\left(\frac{\sin \beta x/2}{\beta/2}\right) F_- - 2\rho \frac{\partial}{\partial\beta}\left(\frac{\sin \beta x/2}{\beta/2}\right)(A^+A)F_-$$
$$+ \rho \frac{\partial}{\partial\beta}\left(\frac{\sin \beta x/2}{\beta/2}\right) F_+ + \mathrm{i}\rho x \left(\frac{\sin \beta x/2}{\beta/2}\right)(A^+A)F_+ - \frac{1}{2}\mathrm{i}\rho x\left(\frac{\sin \beta x/2}{\beta/2}\right) F_-$$
$$\hat{L}_2(x) = \mathrm{i}\rho C(x) F_+^2 + \mathrm{i}\rho D(x)(A^+A) + \mathrm{i}\rho F(x) + \mathrm{i}G(x) \tag{3.9}$$
$$\hat{L}_3(x) = \rho H(x) F_- \qquad \hat{L}_4(x) = \mathrm{i}\rho L(x).$$

The various functions entering the above expression are given below

$$C(x) = -\frac{x}{\beta^2} + \frac{1}{\beta^3}\sin \beta x \qquad D(x) = -10C(x) - 2x\left(\frac{\sin \beta x/2}{\beta/2}\right)^2$$
$$F(x) = -3C(x) - \frac{1}{2}x\left(\frac{\sin \beta x/2}{\beta/2}\right)^2 \qquad G(x) = \beta C(x)$$
$$H(x) = \frac{30}{\beta^4}\sin \beta\frac{x}{2} + \frac{14}{\beta^4}\sin\frac{3}{2}\beta x - \frac{32}{\beta^3}x\cos \beta\frac{x}{2} - \frac{4}{\beta^3}x\cos\frac{3}{2}\beta x$$
$$L(x) = \frac{88}{\beta^2}C(x) - 16\left(\frac{\sin \beta x/2}{\beta/2}\right)^2 C(x) + \frac{16}{\beta^4}x - \frac{4}{\beta^5}\sin 2\beta x. \tag{3.10}$$

We have now further difficulty in writing down the explicit solution we are looking for. Indeed to disentangle the exponents we cannot use the dual of the Weyl–Baker–Hausdorff formula in the form (2.10), but we must take into account a number of successive commutators, in this connection an expansion quoted in the mathematical literature as the Zassenhaus expansion (Witschel 1975) i.e.

$$\begin{aligned}\exp(A+B)=&\exp(A)\exp(B)\exp(-\tfrac{1}{2}[A,B])\\&\times\exp(-\tfrac{1}{3}[[A,B],B]-\tfrac{1}{6}[[A,B],A])\exp(-\tfrac{1}{24}[A,[A,[A,B]]]\\&-\tfrac{1}{8}[B,[A,[A,B]]]-\tfrac{1}{8}[B,[B,[A,B]]])\end{aligned} \tag{3.11}$$

has proved to be very useful.

In this connection, neglecting the terms containing ρ^2, and exploiting again the table of commutators given in the appendix, we arrive at the following expression for the first order perturbed solution of C_l

$$C_l(\tau)=(-\mathrm{i})^l\exp\left(-\frac{\mathrm{i}\alpha}{2}\int_0^\tau \mathrm{d}\tau' N(\tau')\right)\exp\left(-\mathrm{i}\alpha l\frac{\tau}{2}\right)[A_l(\tau)+\mathrm{i}D_l(\tau)] \tag{3.12}$$

where

$$\begin{aligned}A_l(\tau)=&P_l(N(\tau))-\mu\Omega\frac{\partial}{\partial\alpha}\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)[(2l+1)(l+1)^{1/2}P_{l+1}(N(\tau))\\&-(2l-1)\sqrt{l}P_{l-1}(N(\tau))]+\frac{\mu}{\Omega}\left[H(\tau)-\frac{1}{9}\Omega^4\frac{\partial}{\partial\alpha}\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)^3\right]\\&\times[(l+1)^{1/2}P_{l+1}(N(\tau))-\sqrt{l}P_{l-1}(N(\tau))]+\frac{1}{2}\mu\Omega^2\frac{\partial}{\partial\alpha}\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)^2\\&\times\{[(l+1)(l+2)]^{1/2}P_{l+2}(N(\tau))-[l(l-1)]^{1/2}P_{l-2}(N(\tau))\},\\D_l(\tau)=&\frac{1}{2}\mu\Omega\tau\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)[(2l+1)(l+1)^{1/2}P_{l+1}(N(\tau))+(2l-1)\sqrt{l}P_{l-1}(N(\tau))]\\&+\frac{\mu}{\Omega}\left[3\Omega\frac{\sin\alpha\tau/2}{\alpha/2}C(\tau)+2\Omega^4\tau\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)^3\right][(l+1)^{1/2}P_{l+1}(N(\tau))\\&+\sqrt{l}P_{l-1}(N(\tau))]+\frac{\mu}{\Omega}\left[C(\tau)-\frac{1}{2}\Omega^3\tau\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)^2\right]\\&\times\{[(l+1)(l+2)]^{1/2}P_{l+2}(N(\tau))+[l(l-1)]^{1/2}P_{l-2}(N(\tau))\}\\&-4\frac{\mu}{\Omega}\left[2C(\tau)+\Omega^3\tau\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)^2\right]\\&\times lP_l(N(\tau))+\frac{\mu}{\Omega}\left[-2C(\tau)-\Omega^3\tau\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)^2+\frac{88}{\alpha^2}\Omega^2C(\tau)\right.\\&-18\Omega^2\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)^2C(\tau)+\frac{16}{\alpha^4}\Omega^5\tau-\frac{4}{\alpha^5}\Omega^5\sin(2\alpha\tau)\\&\left.-\frac{7}{6}\Omega^5\tau\left(\frac{\sin\alpha\tau/2}{\alpha/2}\right)^4\right]P_l(N(\tau)).\end{aligned} \tag{3.13}$$

$P_l(N(\tau))$ is the Poisson function defined in (2.14) and the other functions of τ entering the above expressions can be easily derived from the corresponding functions of x.

The solution we have found appears to be very complicated, but some further physical assumptions, when one is treating a specific problem, could allow significant simplification in many cases and it is useful to avoid complex numerical calculations when a large number of l are coupled as happens, e.g., in the quantum analysis of the Free Electron Laser (Dattoli and Renieri 1983).

4. Conclusions

One can now ask what is the connection between the RN equation and the HRN one, or whether it is possible to derive them from a more general equation.

We shall discuss this topic within the framework of a specific example.

Let us consider the following equation:

$$\begin{aligned} &\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}C_l(\tau)=\alpha l C_l(\tau)+\Omega[(n+l+1)^{1/2}C_{l+1}(\tau)+(n+l)^{1/2}C_{l-1}(\tau)] \\ &C_l(0)=\delta_{l,0} \end{aligned} \tag{4.1}$$

We have neglected the nonlinear term in l, for the sake of simplicity, but this fact, as it will be seen in the following, by no means diminishes the generality of our argument.

The equation (4.1) resembles the unpertubed HRN equation (2.1) apart from the constant n in the square roots, we shall show how, from the solution (4.1), in specified limits, one can derive both the solution of RN and HRN unperturbed equations (as well as the perturbed ones).

To look for a solution of (4.1) we shall employ the procedure of § 2 with only a few changes.

We first introduce the new 'label'

$$L=n+l \tag{4.2}$$

so that, by performing a transformation analogous to (2.4), we can write down the following equation

$$\begin{aligned} &(\mathrm{d}/\mathrm{d}x)M_L(x)=\hat{T}^0(x)M_L(x) \\ &M_L(0)=(-\mathrm{i})^n\mathrm{i}^L\delta_{L,n} \end{aligned} \tag{4.3}$$

needless to say

$$\begin{aligned} &AM_L=\sqrt{L}M_{L-1} \\ &A^+M_L(L+1)^{1/2}M_{L+1}. \end{aligned} \tag{4.4}$$

Applying now the same procedure as in § 2, we easily find

$$\begin{aligned} C_l(\tau)=(-\mathrm{i})^l\exp(-\mathrm{i}\alpha l\tau/2)\exp\left(-\mathrm{i}\alpha/2\int_0^\tau \mathrm{d}\tau' N(\tau')\right) \\ \times\exp(-N(\tau)/2)[N(\tau)]^{l/2}\sum_{s=0}^{\infty}\frac{(-)^s}{s!(l+s)!}\frac{[(n+l)!\,n!]^{1/2}}{(n-s)!}[N(\tau)]^s. \end{aligned} \tag{4.5}$$

We can now discuss two possible limiting cases:

(i) $n=0$.

This case yields straightforwardly the Poisson function solution already found in § 2.

(ii) $n \gg l$.

This case deserves a few words of comment. The following relations hold

$$[(n+l)!\,n!]^{1/2}/(n-s)! \sim n^{l/2} n!/(n-s)! \sim n^{l/2+s}. \tag{4.6}$$

Furthermore, since

$$\sum_{s=0}^{\infty} \frac{(-)^s}{s!\,(l+s)!} \left(\bar{\Omega} \frac{\sin(\alpha\tau/2)}{\alpha/2}\right)^{l+2s} = J_l\left(2\bar{\Omega} \frac{\sin(\alpha\tau/2)}{\alpha/2}\right) \tag{4.7}$$

where J_l is the lth Bessel function and

$$\bar{\Omega} = \Omega\sqrt{n} \tag{4.8}$$

we find, for $\bar{\Omega} < 1$

$$C_l(\tau) = (-\mathrm{i})^l \exp[-\mathrm{i}\alpha l\tau/2] J_l\left(2\bar{\Omega} \frac{\sin(\alpha\tau/2)}{\alpha/2}\right) \tag{4.9}$$

which is the Bessel type solution already found in Bosco and Dattoli (1983).

The above considerations can also be extended to the perturbed case, the only problem arising is the relative complexity of the algebra involved.

One may now ask whether a more conventional approach to this kind of equation would work.

We will consider for simplicity the equation

$$\mathrm{i}C_l' = \Omega(\sqrt{l+1}\,C_{l+1} + \sqrt{l}\,C_{l-1}) \qquad C_l(0) = \delta_{l,0} \tag{4.10}$$

whose solution within the framework of our formalism may be obtained straightforwardly as

$$C_l(\tau) = (-\mathrm{i})^l [\exp(-\tfrac{1}{2}(\Omega\tau)^2)/\sqrt{l!}](\Omega\tau)^l. \tag{4.11}$$

A more conventional approach to (4.10) may be the one based on the Laplace Transform (LT) technique, indeed the LT of $C_l(\tau)$ is

$$S_l(P) = \int_0^\infty C_l(x)\,\mathrm{e}^{-Px}\,\mathrm{d}x \qquad (x = \Omega t) \tag{4.12}$$

so that inserting (4.12) into (4.10) we find

$$\mathrm{i}pS_l(P) - [(l+1)^{1/2} S_{l+1}(P) + \sqrt{l}\,S_{l-1}(P)] = \mathrm{i}\delta_{l,0} \tag{4.13}$$

which is a linear first order difference equation with variable coefficients, with the important feature that it contains the boundary conditions of the problem.

The solution of (4.13) is not an easy task and we have been unable to find a closed expression by means of a direct technique. However, we can reverse the problem and utilise our technique to give the solution of the difference equation (4.10), indeed (4.11) and (4.13) give

$$S_l(P) = \mathrm{i}^l (1/\sqrt{l!}) \sqrt{\pi/2} (\mathrm{d}/\mathrm{d}p)^l \exp(P^2/2)\,\mathrm{erfc}(P/\sqrt{2}). \tag{4.14}$$

This very simple example may illustrate the potentiality of the method proposed in the present paper. In a forthcoming paper we shall better illustrate the effectiveness of our operatorial technique applying it to differential difference equations of the RN type with time dependent coefficient where the LT approach becomes really impracticable.

The technique we have presented has proved to be very useful for a perturbative analysis of this class of RN-type equations (both harmonic and conventional); we could in principle extend the calculation to higher order in μ, the underlying complexity is only due to the proliferation of commutators both in the Magnus and Zassenhaus expansions. We must, however, emphasise that one can always extract from the exact solution, one of 'practical usefulness' for a specific physical situation, but this needs specific assumption within the framework of the simplification allowed by the problem one is considering.

In a further paper (Dattoli and Richetta 1983) we shall show how the above formalism turns out to be very useful in understanding the coherence properties of the Free Electron Laser and the generation of squeezed states.

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Appendix

In the third section we have introduced the operators

$$F_{\pm} = A^{+}\exp(-\mathrm{i}\beta x/2) \pm A\exp(\mathrm{i}\beta x/2) \tag{A1}$$

whose commutation properties play a crucial role in the procedure of solution we have worked out.

We must emphasise that the various commutators entering our calculations permit, as a remarkable feature, the fact that they can be always expressed in terms of products of (A1) and no extra operators appear.

This kind of 'group' algebraic structure greatly simplifies the machinery of the calculation involved.

In the following we report the table of commutators whose explicit calculation has been required in order to carry out the solution.

$$\begin{aligned}
&[F_{+}, F_{-}] = 2, \qquad [F_{\pm}, A^{+}A] = -F_{\mp}, \qquad [F_{\pm}, A^{+}AF_{\pm}] = -F_{\mp}F_{\pm} \\
&[F_{\pm}, A^{+}AF_{\mp}] = -F_{\mp}^{2} \pm 2A^{+}A, \qquad [F_{\pm}, F_{\pm}F_{\mp}F_{\pm}] = \pm 2F_{\pm}^{2}, \\
&[F_{+}, F_{-}^{2}F_{+}] = -[F_{-}, F_{-}F_{+}^{2}] = 4F_{-}F_{+}, \qquad [F_{\pm}, F_{-}^{3}] = F_{\mp}F_{\pm}^{2}, \\
&[F_{\mp}, F_{\pm}F_{\mp}F_{\pm}] = \mp 4(F_{\mp}F_{\pm} \pm 1) \\
&[F_{\pm}, F_{\pm}^{2}F_{\mp}] = \pm 2F_{\pm}^{2}, \qquad [F_{+}, F_{+}F_{-}^{2}] = -[F_{-}, F_{+}^{2}F_{-}] = 4F_{+}F_{-}
\end{aligned} \tag{A2}$$

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